Keener's method

DSTA

1 Summary of Massey's method

1.1 Massey's vision

Ratings are a unit quantity distributed among tournament participants.

The data that drives ratings is point difference.

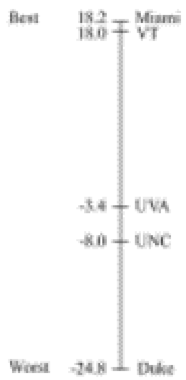
The difference in strenght is latent but revealed by the points difference in a direct match.

By definition, points difference sums to 0; the natural linear algebra formalisation has a singular matrix and is not actionable.

Massey *alters* the matrix to guarantee that a solution exists, if approximate.

Massey's ratings are the solution $\mathbf{r}$ of $\overline{M}\mathbf{r} = \mathbf{p}$

Team	Rating r	Rank
Duke	-24.8	5th
Miami	18.2	1st
UNC	-8.0	4th
UVA	-3.4	3rd
VT	18.0	2nd



2 Keener's method

2.1 Stipulations, 1

One's *strength* should be measured relatively to their opponents'

Team i might be strong against team j but weak against k and so on:

$s_i = \sum_{j=1}^{m} s_{ij}$

where $s_{ii} = 0$ (i cannot play itself)

2.2 Stipulations, 2

As with Massey, ratings are a unit quantity distributed among tournament participants:

$\sum_{i=1}^{m} r_i = 1$

Pie chart effect: one's rating improvement can only come as others' worsens.

Later, ratings will determine rankings and winning probabilities.

2.3 Stipulations, 3

K. believes that strengh, which is manifested, and rating, which is latent, should be connected by a scaling factor λ, which is to be determined for each league/turnament:

$s_i = \lambda r_i$

. . .

So, in vector notation:

$\mathbf{s} = \lambda \mathbf{r}$

At the moment we know neither of the three… let's start with strenght.

2.4 The input data

K. does not commit to a specific way to gauge strength:

a_{ij} = the statistics produced by team i when playing j

non-negativity requirement: $a_{ij} \geq 0$

2.5 Example Stats - A

Consider wins/ties:

$$a_{ij} = W_{ij} + \frac{T_{ij}}{2}$$

2.6 Example Stats - B

Points scored against:

$a_{ij} = S_{ij}$

Points is considered a *crude* measure of strength.

Avoid high-scoring matches to have a disproportionate effect by means of relative scoring:

$$a_{ij} = \frac{S_{ij}}{S_{ij} + S_{ji}}$$

2.7 Laplace correction

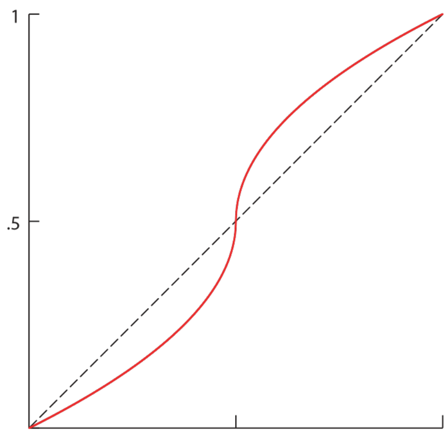
$$a_{ij} = \frac{S_{ij} + 1}{S_{ij} + S_{ji} + 2}$$

. . .

if $S_{ij} \approx S_{ji}$ and both are large then $a_{ij} \approx \frac{1}{2}$ (Good or bad?)

2.8 Skewing

- it mitigates convergence to $\frac{1}{2}$ over time
- it sterilises the effect of exteme scores



. . .

$$h(x) = \frac{1}{2} + \text{sgn}\{x - (1/2)\}\sqrt{|2x - 1|}/2$$

additionally, $a_{ij} \leftarrow \frac{a_{ij}}{n_i}$ to balance no. of games.

2.9 keener's strength

Strenght revealed by performance (scoring) but tempered by the strength of the opponent themselves.

Relative s. of *i* when playing against *j*:

$$s_{ij} = a_{ij} \cdot r_j$$

(N.B. *scoring* is S_{ij} while *strength* is s_{ij})

2.10 Cumulative strenght

Cumulative/absolute strenght of team *i:*

$$s_i = \sum_{j=1}^{m} s_{ij}$$

$$\mathbf{s} = \begin{pmatrix} \sum_{j=1}^{m} s_{1j} \\ \sum_{j=1}^{m} s_{2j} \\ \vdots \\ \sum_{j=1}^{m} s_{mj} \end{pmatrix}$$

$$\mathbf{s} = \begin{pmatrix} \sum_{j=1}^{m} s_{1j} \\ \sum_{j=1}^{m} s_{2j} \\ \vdots \\ \sum_{j=1}^{m} s_{mj} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{pmatrix}$$

$$\mathbf{s} = \begin{pmatrix} \sum_{j=1}^{m} s_{1j} \\ \sum_{j=1}^{m} s_{2j} \\ \vdots \\ \sum_{j=1}^{m} s_{mj} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{pmatrix} = A\mathbf{r}$$

. . .

The *strength vector* $\mathbf{s}$ that collects all cumulative strengths is

$\mathbf{s} = A\mathbf{r}$

where $\mathbf{r}^T = \{r_1, \dots r_m\}$ is the rating vector.

The argument has a certain circularity...

2.11 Finally

Since rating should be proportional to strength:

$$\mathbf{s} = \lambda\mathbf{r}$$

. . .

$$A\mathbf{r} = \lambda\mathbf{r}$$

So, rating really is an e-vector of A, and λ an e-value.

2.12 Observations

We would like a positive λ

also the values in $\mathbf{r}$ should be positive

. . .

In general, a *reasonable* solution is **not** guaranteed:

- which eigenvalue (among up to m) to choose?
- even for positive λs the relative e-vector could contain negative or even complex numbers!

3 The Perron-Frobenius theorem

3.1 Non-negativity

Perron-Frobenius focus on matrices that contain only non-negative values:

$A = [a_{ij}] \geq 0$

This is easily the case when a_{ij} is a statistic on winning or scoring etc.

. . .

3.2 Irreducibility

P-F request that each pair i, j be *connected:*

- simply, $a_{ij} > 0$ (i.e., teams have played before)
- or there is a non-negative path of p intermediate "steps" $k_1, \ldots k_p$:

$$a_{ik_1} > 0, a_{k_1 k_2} > 0, \ldots a_{k_p j} > 0$$

3.3 Irreducibility in practice

it requiring that each teams has played common opponents in the past, even indirectly, e.g.:

$a_{\text{Burnley,Nice}} = 0$

but since

$$a_{\text{Burnley,Arsenal}} > 0, a_{\text{Arsenal,PSG}} > 0, a_{\text{PSG,Nice}} > 0$$

a tournament containing both Burnley and Nice is suitable.

Irred. may not hold at the beginning of a tournament but it's not considered **prohibitive.**

3.4 Good news

If A is non-negative and irreducible, then

- the dominant e-value is real and strictly positive: our λ!
- except for positive multiples, there's only one non-negative e-vector $\mathbf{x}$ for A: (almost) our $\mathbf{r}$!
- the final $\mathbf{r}$ is obtained by normalizing $\mathbf{x}$: $\mathbf{r} = \mathbf{x}/\sum_j x_j$
- individual ratings r_i will be in (0,1) and will sum to 1.

3.5 Perron-Frobenius

Perron–Frobenius Theorem

If $\mathbf{A}_{m\times m} \geq \mathbf{0}$ is irreducible, then each of the following is true.

- Among all values of λ_i and associated vectors $\mathbf{x}_i \neq \mathbf{0}$ that satisfy $\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i$ there is a value λ and a vector $\mathbf{x}$ for which $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ such that

  ▷ λ is real. ▷ $\lambda > 0$.
  ▷ $\lambda \geq |\lambda_i|$ for all i. ▷ $\mathbf{x} > 0$.

- Except for positive multiples of $\mathbf{x}$, there are no other nonnegative eigenvectors $\mathbf{x}_i$ for $\mathbf{A}$, regardless of the eigenvalue λ_i.
- There is a unique vector $\mathbf{r}$ (namely $\mathbf{r} = \mathbf{x}/\sum_j x_j$) for which

$$\mathbf{A}\mathbf{r} = \lambda\mathbf{r}, \quad \mathbf{r} > \mathbf{0}, \quad \text{and} \quad \sum_{j=1}^{m} r_j = 1. \tag{4.11}$$

- The value λ and the vector $\mathbf{r}$ are respectively called the ***Perron value*** and the ***Perron vector.*** For us, the Perron value λ is the proportionality constant in (4.9), and the unique Perron vector $\mathbf{r}$ becomes our ***ratings vector.***

3.6 Observations

- the conditions are strict but not impossible
- a strong memory effect makes Keener's ratings represent long-term tendencies
- today, random walks/Montecarlo methods approximate Keener's rating without the need to extract e-pairs of large matrices.

- [Keener, SIAM Review 35:1, March 1993] is credited with seeding the ideas behind Google's PageRank.