# Math concepts for Data Science

AP

## From datasets to matrices

### Motivations

Activity tables show how users *map* their choices or, viceversa, how available products *map* onto their adopters.

| | Matrix | Alien | Star Wars | Casablanca | Titanic |
|---|---|---|---|---|---|
| Joe | 1 | 1 | 1 | 0 | 0 |
| Jim | 3 | 3 | 3 | 0 | 0 |
| John | 4 | 4 | 4 | 0 | 0 |
| Jack | 5 | 5 | 5 | 0 | 0 |
| Jill | 0 | 0 | 0 | 4 | 4 |
| Jenny | 0 | 0 | 0 | 5 | 5 |
| Jane | 0 | 0 | 0 | 2 | 2 |

Figure 11.6: Ratings of movies by users

Running example from Ch. 11, p. 430 of MMDS.

### From tables to matrices

$$A_{7\times 5} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 2 & 2 \end{pmatrix}$$

. . .

In Lin. Algebra the matrix *"forgets"* the labels for rows/cols., e.g., Joe/1st row, The Matrix/1st col., Alien/2nd col. etc.

### 1-hot encondings

$$B_{18\times 13} = \left|\begin{array}{cccc|ccc|c} 1 & 0 & 0 & \ddots & 1 & 0 & \ddots & 1 \\ 1 & 0 & 0 & \ddots & 0 & 1 & \ddots & 1 \\ 1 & 0 & 0 & \ddots & & \ddots & & \vdots \\ & & & \ddots & & \ddots & & \vdots \end{array}\right.$$

1st col. indicates that Joe watched the film

8th col indicates that The Matrix was the film watched

the final col. is views (or ratings) from the original table: 18 reviews overall.

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$$B_{18\times 13} = \left|\begin{array}{cccc|ccc|c} 1 & 0 & 0 & \ddots & 1 & 0 & \ddots & 1 \\ 1 & 0 & 0 & \ddots & 0 & 1 & \ddots & 1 \\ 1 & 0 & 0 & \ddots & & \ddots & & \vdots \\ & & & \ddots & & \ddots & & \vdots \end{array}\right.$$

$U \cdot F \cdot \mathbf{r}$ ( where $\cdot$ means concatenation)

## Data Science as Linear Algebra

### Linear equations

*Q:* given a user's declared appreciation of Science fiction, how could it be imputed to the films they have reviewed?

. . .

A system of linear equations:

$$a_{i_1} x_1 + a_{i_2} x_2 + ... a_{i_n} x_n = b_i$$

. . .

$$A\mathbf{x} = \mathbf{b}$$

(we use **r** instead of **b** to remember that those are *ratings*)

. . .

Interpretation: how each film contributed to determine this user's appreciation for the Sci-Fi genre.

## Data Science as Geometry

### In Mathematics

a matrix represents a linear transformation, a particular type of *mapping,* between two (linear) spaces.

It is just one of the possible representations of a mapping -it depends on a choice for the bases for source and target spaces.

. . .

Now we can apply the full machinery of Linear Algebra/Geometry and see what happens.

We apply linear maps (in particular, eigenvalues and eigenvectors) to matrices that *do not represent geometric transformations,* but rather some kind of relationship between entities (e.g., users and films).

### datapoints are vectors

A user experience is represented by a vector: user's ratings for each film. E.g.,

**joe** = <1, 1, 1, 0, 0>

**jill** = <0, 0, 0, 4, 4>

These are *row vectors* while normally vectors are columns. The transpose $T$ operator inverts row and colums: $\mathbf{joe}^T$ is a column vector.

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$$\mathbf{joe}^T = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

. . .

$$A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

---

*Q:* Can the given users' experiences be combined to yield a specific point $\mathbf{r}$ that represents a rating of how much each user likes Sci-Fi?

$\mathbf{r} = < 6, 9, 10, .5, 1 >^T$

. . .

### Independent vectors

Geometry sees vectors (user experiences) as axes of a reference system that *spans* a space of possible ratings.

That is possible only if at least n vectors are independent from each other.

That is automatically the case for the axes of a Cartesian diagram, or for any set of *orthogonal* vectors.

### Dependent vectors

Two vectors are dependent when one is simply a multiple of the other: their direction is the same but for *stretching* or *compression.*

Dependent v. should be detected and, if possible, excluded.

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Non-independence example: I only watch Jason Bourne films at my friend's

U = {Alb, Ale}, F = {The-B-id, The-B-ultimatum, …}

$$A_{Bourne} = \begin{pmatrix} 4 & 4 & 4 & 0 & 2 \\ 2 & 2 & 2 & 0 & 1 \end{pmatrix}$$

The two rows are dependent! Ale depends on Alb for watching Jason Bourne films.

---

Test: can you find two numbers $x_1$ and $x_2$ s.t.

$x_1 \cdot \mathbf{alb}^T + x_2 \cdot \mathbf{ale}^T = \mathbf{0}$?

(here $\cdot$ means multiplication)

. . .

Simplest solution: $x_1 = 1$ and $x_2 = -2$.

## Background: determinant

The determinant understand the matrix as an area

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

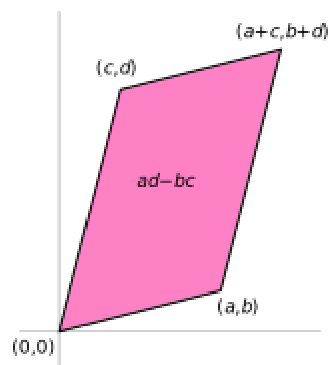


Figure 1: area

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if the determinant of $A$ is zero, $|A| = 0$, then

- column vectors are not independent;
- $A$ does not have a unique inverse matrix, so
- **it is not amenable to further processing.**

. . .

## Matrix rank

The *Rank* of a matrix A is the dimension of the vector space generated by its columns. It corresponds to

1. the maximum number of linearly-independent columns
2. the dimension of the space spanned by the rows

We consider data matrices with independent columns, ie., $rank(A_{m\times n}) = n$.

## Matrix inversion, I

The identity matris $I$ (or $U$) is the unit matrix: $I \cdot I = I$.

$$I = \begin{pmatrix} 1 & 0 & \dots & \\ 0 & 1 & & \\ \vdots & & \ddots & \end{pmatrix}$$

. . .

```
import numpy as np

myI = np.eye(n)
```

creates the square identity matrix of size n.

## Matrix inversion, II

Given A, find its (left) inverse $A^{-1}$ s.t.

$A^{-1} \cdot A = I$

. . .

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

. . .

$$A^{-1} = \begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix}$$

---

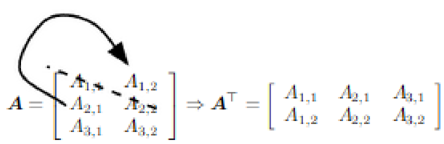
Matrix inversion is a delicate process:

- The inverse may not exist, or be non-unique.
- it might have numerical issues, so $A^{-1} \cdot A$ only $\approx I$.

```
print(Ainverse.dot(A))

[[1.00000000e+00 0.00000000e+00]
 [1.11022302e-16 1.00000000e+00]]
```

---

Inversion is only defined for square matrices, so if $A$ is not square we then use the square matrix $A' = A^T \cdot A$



{:height="50%" width="50%"}

. . .

```
import numpy as np

a = np.array([[11, 12], [21, 22],  [31, 32]])

at2 = a.transpose()
```

## Computing

### Numpy

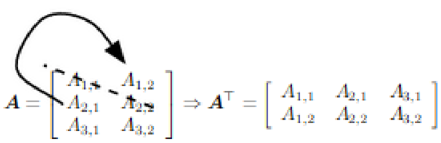
Numpy extends Python to numerical computation.

To handle large data it creates view rather copies of arrays/matrices.

```python
import numpy as np

a = np.array([[11, 12], [21, 22],  [31, 32]])

# changeable array
at2 = a.transpose()

a[0][0] = 111

print(a)

print(at2)
```

---

Alternative transposition:



```python
import numpy as np

a = np.array([[11, 12], [21, 22],  [31, 32]])

# tuples, not lists, and only once
at = zip(*a)

for row in at:
    print(row)
```

### Matrix multiplication

```python
A.dot(B) == A @ B # matrix multiplication

A.dot(B) != A * B # element-wise product
```

`@` generalises to *tensors:* three-dimensional matrices.

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```
m = np.array([[4, 4, 4, 0, 2], [2, 2, 2, 0, 1]])

mt = m.transpose()

mprime = m.dot(mt)

print(mprime.shape)
```

Here (the Jason Bourne ex.) rows are *not* independent. This is revealed by $|M'| = 0$

```
print(np.linalg.det(mprime))

0.0
```

---

Matrix inversion and checking for errors in the results

```
i = np.eye(2)

if (np.linalg.det(mprime)):

 mprime_inv =  np.linalg.inv(mprime)

 mprime_dot_mprime_inv = mprime.dot(mprime_inv)

 # handles inf and tiny vals
 print(np.allclose(mprime_dot_mprime_inv, i))
```

prints `true` if `mprime_dot_mprime_inv` is element-wise equal to `i` within a *tolerance.*